METRIC AND TOPOLOGICAL SPACES: EXAM 2024/25

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Problem 1 (10%). If d is a metric on $\mathcal{X} \ni x, y$, will $\varrho(x,y) := \ln(1 + \mathrm{d}(x,y))$ also be ? (state and prove)

Bonus (*b*) (25%). If $(\mathcal{X}, \varrho)$ is complete, was $(\mathcal{X}, \mathrm{d})$ also complete ? • If $(\mathcal{X}, \mathrm{d})$ was complete, will $(\mathcal{X}, \varrho)$ also be ?

Problem 2 (10+15%). In every metric space $(\mathcal{X}, \mathrm{d})$, the intersection of all open sets containing a point x is the singleton set $\{x\}$. (prove)

(*b*) If the intersection of all open subsets of $\mathcal{X}$ containing x is the singleton set $\{x\}$, then $\mathcal{X}$ need not be Hausdorff. (give an example)

Problem 3 (20%). A subset $A \subseteq \mathcal{X}$ of a space $\mathcal{X}$ is such that $\forall\, x, y \in A$ there exists a connected set $B(x,y) \subseteq A$ with $B \ni x, y$. Prove A is connected.

Problem 4 (30%). Suppose that a metric space $(\mathcal{X}, \mathrm{d})$ is such that every continuous function $f\colon \mathcal{X} \to \mathbb{E}^1$ is bounded. Prove that $\mathcal{X}$ is (sequentially) compact.

Problem 5 (15%). Give an example: a metric space $(\mathcal{X}, \mathrm{d})$ is complete and $f\colon \mathcal{X} \to \mathcal{X}$ is such that $\mathrm{d}(f(x), f(y)) < \mathrm{d}(x,y)$ for all $x \neq y$ in $\mathcal{X}$, but f has no fixed point.

Date: November 4, 2024. Good luck !